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APPROXIMATION FOR OPERATOR RICCATI EQUATIONS

I. G. Rosen

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ON HILBERT-SCHMIDT NORM CONVERGENCE OF GALERKIN APPROXIMATION FOR OPERATOR RICCATI EQUATIONS

I. G. Rosen¹
Department of Mathematics
University of Southern California
Los Angeles, CA 90089

ABSTRACT

An abstract approximation framework for the solution of operator algebraic Riccati equations is developed. The approach taken is based upon a formulation of the Riccati equation as an abstract nonlinear operator equation on the space of Hilbert-Schmidt operators. Hilbert-Schmidt norm convergence of solutions to generic finite dimensional Galerkin approximations to the Riccati equation to the solution of the original infinite dimensional problem is argued. The application of the general theory is illustrated via an operator Riccati equation arising in the linear-quadratic design of an optimal feedback control law for a one dimensional heat/diffusion equation. Numerical results demonstrating the convergence of the associated Hilbert-Schmidt kernels are included.

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I.G. ROSEN

Abstract. An abstract approximation framework for the solution of operator algebraic Riccati equations is developed. The approach taken is based upon a formulation of the Riccati equation as an abstract nonlinear operator equation on the space of Hilbert-Schmidt operators. Hilbert-Schmidt norm convergence of solutions to generic finite dimensional Galerkin approximations to the Riccati equation to the solution of the original infinite dimensional problem is argued. The application of the general theory is illustrated via an operator Riccati equation arising in the linear-quadratic design of an optimal feedback control law for a one dimensional heat/diffusion equation. Numerical results demonstrating the convergence of the associated Hilbert-Schmidt kernels are included.

1. INTRODUCTION

In this paper we develop an abstract approximation theory for algebraic Riccati equations on spaces of Hilbert-Schmidt operators. Our approach is based upon Barbu's [1] formulation of a class of Riccati equations as abstract nonlinear operator equations on a space of Hilbert-Schmidt operators. We argue that solutions to generic finite dimensional Galerkin approximations to the Riccati equation converge in Hilbert-Schmidt norm to the solution of the original infinite dimensional equation.

Our effort here is closely related to results in one of our earlier papers [6] wherein we developed an approximation theory for operator Riccati differential equations using techniques similar to those which will be employed below. Our treatments here and in [6] differ from the standard approach to the analysis of operator Riccati equation approximation in that we consider the nonlinear operator equations directly in the space of Hilbert-Schmidt operators rather than integral equation equivalents and their limiting properties as the time horizon tends to infinity (see, for example, [3]). While we do in fact obtain a stronger convergence result than the

We seek a solution $\Pi \in \mathcal{C}$ to (2.3). We note that for n a positive integer, the operator \mathcal{B} given by $\mathcal{B}(\Phi) = \Phi^n$, $\Phi \in \mathcal{C}$ (i.e. $f(z) = z^n$) can be shown to satisfy the conditions above. Indeed, boundedness follows from the estimate $|\Phi^n|_{\mathcal{H}} \leq |\Phi|_{\mathcal{H}}^n$, while monotonicity can be established via

$$[\Phi^n - \Psi^n, \Phi - \Psi]_{\mathcal{H}} = \sum_{j=1}^n [\Phi^{n-j} \{\Phi - \Psi\} \Psi^{j-1}, \Phi - \Psi]_{\mathcal{H}} \geq 0,$$

for $\Phi, \Psi \in \mathcal{C}$ (see [9]). Finally, for $\Psi \in \mathcal{C}$ and $\lambda > 0$ let $\{\psi_i\}_{i=1}^{\infty}$ be the orthonormal set of eigenvectors of Ψ with corresponding eigenvalues $\{\alpha_i\}_{i=1}^{\infty}$. (The fact that $\Psi \in \mathcal{C}$ implies $\alpha_i \geq 0$, $i = 1, 2, \dots$). Then, if we let γ_i denote a nonnegative solution to the equation $\gamma_i + \lambda \gamma_i^n - \alpha_i = 0$ (the intermediate value theorem guarantees that such a γ_i exists with $0 \leq \gamma_i \leq \alpha_i$) and set $\Phi\varphi = \sum_{i=1}^{\infty} \gamma_i(\varphi, \psi_i)\psi_i$, $\varphi \in H$, it follows that $\Phi \in \mathcal{C}$ and $\Phi + \lambda\Phi^n = \Psi$. This establishes (2.2). When $n = 2$, (2.3) becomes the familiar quadratic Riccati equation.

Define the operator $\mathcal{A} \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*)$ by

$$\mathcal{A}\Phi = A^*\Phi + \Phi A, \quad \Phi \in \mathcal{V}.$$

It can be argued (see [1]) that \mathcal{A} is strongly \mathcal{V} -elliptic - that is there exists a constant $\omega > 0$ for which

$$(2.4) \quad [\mathcal{A}\Phi, \Phi]_{\mathcal{H}} \geq \omega \|\Phi\|_{\mathcal{V}}^2, \quad \Phi \in \mathcal{V}.$$

If we define the subspace $Dom(\mathcal{A}) = \{\Phi \in \mathcal{V} : \mathcal{A}\Phi \in \mathcal{H}\}$ then it follows (see [8]) that the operator $\mathcal{A} : Dom(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is densely defined and m -accretive in \mathcal{H} . With the above definitions the problem of finding a solution to the operator algebraic Riccati equation (2.3) becomes one of finding a solution $\Pi \in Dom(\mathcal{F})$ to the abstract nonlinear operator equation in \mathcal{H} given by

$$(2.5) \quad \mathcal{F}(\Pi) = \Theta$$

where $\Theta \in \mathcal{C}$ is given and $\mathcal{F} : Dom(\mathcal{F}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is the operator defined by $\mathcal{F}(\Phi) = \mathcal{A}\Phi + \mathcal{B}(\Phi)$ for $\Phi \in Dom(\mathcal{F}) = \mathcal{C} \cap Dom(\mathcal{A})$.

Using a standard fixed point argument on the closed convex subset \mathcal{C} , Barbu [1] argues that the equation

$$\mathcal{F}_{\lambda}(\Phi_{\lambda}) = \Theta$$

has a unique solution $\Phi_{\lambda} \in Dom(\mathcal{F})$ for each $\lambda > 0$ where $\mathcal{F}_{\lambda} : Dom(\mathcal{F}) \subset \mathcal{H} \rightarrow \mathcal{H}$ is the Yosida-like approximation to \mathcal{F} given by $\mathcal{F}_{\lambda} = \mathcal{A} + \lambda^{-1}\{I - (I + \lambda\mathcal{B})^{-1}\}$. Then using the boundedness and monotonicity of \mathcal{B} , Barbu argues further that the Π_{λ} converge in \mathcal{H} as $\lambda \rightarrow 0$ to an operator $\Pi \in Dom(\mathcal{F})$ which is a solution to (2.5) (or, equivalently, (2.3)). The strong \mathcal{V} -ellipticity of \mathcal{A} (i.e. (2.4)), and the monotonicity

of \mathcal{B} (i.e. (2.1)) are then used in the usual way to establish the uniqueness of Π . Note that $\Pi \in \text{Dom}(\mathcal{F})$ implies that $\Pi \in \mathcal{C}$, i.e. that it is a nonnegative, self-adjoint operator in \mathcal{H} , and that $\Pi \in \mathcal{V}$ with $\mathcal{A}\Pi = A^*\Pi + \Pi A \in \mathcal{H}$.

3. GALERKIN APPROXIMATION AND CONVERGENCE THEORY

For each $n = 1, 2, \dots$ let H_n be a finite dimensional subspace of H with $H_n \subset V$, $n=1, 2, \dots$. Let $P_n : H \rightarrow H_n$ be the orthogonal projection of H onto H_n with respect to the (\cdot, \cdot) inner product on H . We assume

$$(3.1) \quad \lim_{n \rightarrow \infty} \|P_n \varphi - \varphi\| = 0, \quad \varphi \in V.$$

Note that assumption (3.1) implies that $\lim_{n \rightarrow \infty} |P_n \varphi - \varphi| = 0$, $\varphi \in H$, and that the P_n are uniformly bounded in the uniform operator topologies on $\mathcal{L}(H)$ and $\mathcal{L}(V)$.

Lemma 3.1 The operators P_n admit extensions, which we shall again call P_n , to idempotent, uniformly bounded operators in $\mathcal{L}(V^*)$. Moreover, $\lim_{n \rightarrow \infty} \|P_n \varphi - \varphi\|_* = 0$, $\varphi \in V^*$, the V^* -adjoint (i.e. the operator P_n^* satisfying $\langle P_n \varphi, \psi \rangle_* = \langle \varphi, P_n^* \psi \rangle_*$, $\varphi, \psi \in V^*$) is given by $P_n^* = \gamma P_n \gamma^{-1}$, and $\lim_{n \rightarrow \infty} \|P_n^* \varphi - \varphi\|_* = 0$, $\varphi \in V^*$.

Proof For $\varphi \in V^*$ set $P_n \varphi = \varphi_n$ where φ_n is the representer of the functional on H_n which is the restriction of φ to H_n . That is $(\varphi_n, \psi_n) = (P_n \varphi, \psi_n) = (\varphi, \psi_n)$, $\psi_n \in H_n$. It is clear that P_n as given above is a well defined extension of the orthogonal projection of H onto H_n and that it is idempotent. Moreover, since $H_n \subset V \subset V^*$, for $\varphi \in V^*$ we may consider $P_n \varphi = \varphi_n \in H_n$ an element in V^* via the duality pairing $(P_n \varphi, \psi) = (\varphi_n, \psi)$, $\psi \in V$. Then for $\varphi \in V^*$, $\psi \in V$, and $\varphi_n = P_n \varphi$ we have $(P_n \varphi, \psi) = (\varphi_n, \psi) = (\varphi_n, P_n \psi) = (\varphi, P_n \psi)$. Consequently for $\varphi \in V^*$ we have

$$\|P_n \varphi\|_* = \sup_{\substack{\psi \in V \\ \|\psi\| \leq 1}} |(P_n \varphi, \psi)| = \sup_{\substack{\psi \in V \\ \|\psi\| \leq 1}} |(\varphi, P_n \psi)| \leq \|\varphi\|_* \|P_n\|,$$

or $\|P_n\|_* \leq \|P_n\|$. Thus assumption (3.1) implies that the P_n are uniformly bounded in $\mathcal{L}(V^*)$. (We note that alternatively the same extension of the projections P_n to operators on V^* could have been obtained by using the standard approach based upon the density of H in V^* and the usual extension construction in terms of limits.) Now for $\varphi \in H$ we have

$$\begin{aligned} \langle P_n \varphi, \psi \rangle_* &= (P_n \varphi, \gamma^{-1} \psi) = (\varphi, P_n \gamma^{-1} \psi) = (\varphi, \gamma^{-1} \gamma P_n \gamma^{-1} \psi) \\ &= \langle \varphi, \gamma P_n \gamma^{-1} \psi \rangle_*, \end{aligned}$$

and consequently that the V^* -adjoint of P_n , P_n^* , is given by $P_n^* = \gamma P_n \gamma^{-1}$. Finally, assumption (3.1) yields

$$\lim_{n \rightarrow \infty} \|P_n^* \varphi - \varphi\|_* = \lim_{n \rightarrow \infty} \|\gamma^{-1} P_n^* \varphi - \gamma^{-1} \varphi\|$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \|\gamma^{-1} \gamma P_n \gamma^{-1} \varphi - \gamma^{-1} \varphi\| \\ &= \lim_{n \rightarrow \infty} \|P_n \gamma^{-1} \varphi - \gamma^{-1} \varphi\| = 0 \end{aligned}$$

for each $\varphi \in V^*$ and the lemma is proved.

Define the sequence of finite dimensional subspaces $\mathcal{H}_n, n = 1, 2, \dots$ of \mathcal{H} by

$$\mathcal{H}_n = \{\Phi_n P_n : \Phi_n \in \mathcal{L}(H_n)\}.$$

Clearly H_n finite dimensional implies that all operators in \mathcal{H}_n are of finite rank and thus that \mathcal{H}_n is a subspace of both \mathcal{H} and \mathcal{V} . For each $n = 1, 2, \dots$ let $\mathcal{C}_n \subset \mathcal{H}_n$ be the closed convex cone given by $\mathcal{C}_n = \{\Phi_n P_n \in \mathcal{H}_n : \Phi_n = \Phi_n^*, \Phi_n \geq 0\}$. Note that $\mathcal{C}_n \subset \mathcal{C}, n = 1, 2, \dots$. We define Galerkin approximations to the operator $\mathcal{F}, \mathcal{F}_n : \text{Dom}(\mathcal{F}_n) \subset \mathcal{H}_n \rightarrow \mathcal{H}_n$, as follows:

$$(3.2) \quad \mathcal{F}_n(\Phi_n P_n) = \{\mathcal{A}\Phi_n P_n + \mathcal{B}(\Phi_n P_n)\}|_{\mathcal{H}_n}, \quad \text{for } \Phi_n P_n \in \text{Dom}(\mathcal{F}_n) = \mathcal{C}_n.$$

That is, for $\Phi_n P_n \in \mathcal{C}_n$, $\mathcal{F}_n(\Phi_n P_n) = \Psi_n P_n \in \mathcal{H}_n$ where $\Psi_n P_n$ is that element in \mathcal{H}_n (guaranteed to exist and be unique by the Riesz Representation Theorem applied to the functional $\mathcal{A}\Phi_n P_n + \mathcal{B}(\Phi_n P_n) \in \mathcal{V}^*$ restricted to a functional on the finite dimensional Hilbert space \mathcal{H}_n) which satisfies $[\mathcal{A}\Phi_n P_n + \mathcal{B}(\Phi_n P_n), \chi_n P_n]_{\mathcal{H}} = [\Psi_n P_n, \chi_n P_n]_{\mathcal{H}}, \chi_n P_n \in \mathcal{H}_n$.

It is of some value to note that the approximations to \mathcal{F} given in (3.2) are the same ones that would be obtained via the standard approach which is based upon the replacement of the operators A and B in (2.3) by their respective Galerkin approximations on H_n and \mathcal{H}_n . Indeed, for each $n = 1, 2, \dots$ define the operators $A_n \in \mathcal{L}(H_n)$ and $B_n : \mathcal{C}_n \subset \mathcal{H}_n \rightarrow \mathcal{H}_n$ by $A_n \varphi_n = \psi_n$, where for $\varphi_n \in H_n$, ψ_n is that element in H_n which satisfies $(\mathcal{A}\varphi_n, \chi_n) = (\psi_n, \chi_n), \chi_n \in H_n$, and $B_n(\Phi_n P_n) = P_n \mathcal{B}(\Phi_n P_n) P_n$. From the fact that P_n is the orthogonal projection of H onto H_n it follows that $[\Phi P_n, \Psi_n]_{\mathcal{H}} = [\Phi, \Psi_n]_{\mathcal{H}}$ for every $\Phi \in \mathcal{V}^*$ and $\Psi_n \in \mathcal{H}_n$. Then, for $\Phi_n P_n, \Psi_n P_n \in \mathcal{H}_n$ we have

$$\begin{aligned} [\mathcal{F}_n(\Phi_n P_n), \Psi_n P_n]_{\mathcal{H}} &= [\mathcal{A}\Phi_n P_n + \mathcal{B}(\Phi_n P_n), \Psi_n P_n]_{\mathcal{H}} \\ &= [\{\mathcal{A}\Phi_n P_n + \mathcal{B}(\Phi_n P_n)\} P_n, \Psi_n P_n]_{\mathcal{H}} \\ &= [A^* \Phi_n P_n + \Phi_n P_n A P_n + \mathcal{B}(\Phi_n P_n) P_n, \Psi_n P_n]_{\mathcal{H}} \\ &= \sum_{k=1}^{\infty} \{ (A^* \Phi_n P_n e_k, \Psi_n P_n e_k) + (\Phi_n P_n A P_n e_k, \Psi_n P_n e_k) \\ &\quad + (\mathcal{B}(\Phi_n P_n) P_n e_k, \Psi_n P_n e_k) \} \\ &= \sum_{k=1}^{\infty} \{ (A_n^* \Phi_n P_n e_k, \Psi_n P_n e_k) + (A_n P_n e_k, \Phi_n^* \Psi_n P_n e_k) \\ &\quad + (P_n \mathcal{B}(\Phi_n P_n) P_n e_k, \Psi_n P_n e_k) \} \\ &= \sum_{k=1}^{\infty} (\{ A_n^* \Phi_n + \Phi_n A_n + \mathcal{B}_n(\Phi_n P_n) \} P_n e_k, \Psi_n P_n e_k), \end{aligned}$$

or,

$$\mathcal{F}_n(\Phi_n P_n) = \{A_n^* \Phi_n + \Phi_n A_n + \mathcal{B}_n(\Phi_n P_n)\} P_n.$$

In the particular case when $\mathcal{B}(\Phi) = \Phi^2$, for example, the operators \mathcal{F}_n take the form $\mathcal{F}_n(\Phi_n P_n) = \{A_n^* \Phi_n + \Phi_n A_n + \Phi_n^2\} P_n$.

Set $\Theta_n = P_n \Theta P_n \in \mathcal{C}_n$ and consider the problem of finding a solution $\Pi_n \in \mathcal{C}_n$ to the nonlinear operator equation

$$(3.3) \quad \mathcal{F}_n(\Pi_n) = \Theta_n$$

in \mathcal{H}_n . Arguments similar to those described in section 2 above yield that for each $n = 1, 2, \dots$, the equation (3.3) admits a unique solution $\Pi_n \in \mathcal{C}_n$. We shall argue that $\lim_{n \rightarrow \infty} \|\Pi_n - \Pi\|_{\mathcal{V}} = 0$; that is, that the Π_n converge in the \mathcal{V} -Hilbert-Schmidt norm to the solution Π to the equation (2.5) (or equivalently (2.3)). In order to do this we shall require the following lemmas.

Lemma 3.2 If $\{a_i\}_{i=1}^{\infty}$ is an absolutely summable sequence of real numbers, then there exist sequences $\{b_i\}_{i=1}^{\infty}$ and $\{c_i\}_{i=1}^{\infty}$ for which $\lim_{i \rightarrow \infty} b_i = 0$, $\{c_i\}_{i=1}^{\infty}$ is absolutely summable, and $a_i = b_i c_i$, $i = 1, 2, \dots$

Proof Set $\alpha = \sum_{i=1}^{\infty} |a_i|$ and, for $j = 0, 1, 2, \dots$ define the nonnegative integers k_j by $k_0 = 0$ and for $j = 1, 2, \dots$ let k_j be the first index for which

$$\sum_{i=1}^{k_j} |a_i| > \alpha - \frac{1}{j^3}.$$

Then setting $b_i = 1/j$ and $c_i = j a_i$, for $i = k_{j-1} + 1, \dots, k_j$, $j = 1, 2, \dots$, we have $b_i c_i = a_i$, $i = 1, 2, \dots$, $\lim_{i \rightarrow \infty} b_i = 0$, and

$$\begin{aligned} \sum_{i=1}^{\infty} |c_i| &= \sum_{j=1}^{\infty} j \sum_{k=k_{j-1}+1}^{k_j} |a_k| = \sum_{j=1}^{\infty} j \left\{ \sum_{k=1}^{k_j} |a_k| - \sum_{k=1}^{k_{j-1}} |a_k| \right\} \\ &\leq \sum_{k=1}^{k_1} |a_k| + \sum_{j=2}^{\infty} j \left\{ \alpha - \left(\alpha - \frac{1}{(j-1)^3} \right) \right\} \\ &\alpha + \sum_{j=1}^{\infty} \frac{1}{j^2} + \sum_{j=1}^{\infty} \frac{1}{j^3} < \infty \end{aligned}$$

Lemma 3.3 Let X and Y be real separable Hilbert spaces with inner products denoted by $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ respectively. Then every $\Phi \in HS(X, Y)$ can be written in factored form as $\Phi = \Phi^1 \Phi^2$ with $\Phi^1 \in \mathcal{L}(Y)$ compact and $\Phi^2 \in HS(X, Y)$.

Proof Let $\{x_i\}_{i=1}^{\infty}$ be an orthonormal basis for X and let $\{y_i\}_{i=1}^{\infty}$ be an orthonormal basis for Y . If $\Phi \in HS(X, Y)$ then it has a representation in the form of an infinite

matrix $\Phi \leftrightarrow [\varphi_{ij}]$ where $\varphi_{ij} = \langle y_i, \Phi x_j \rangle_Y$, and $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varphi_{ij}^2 < \infty$. Now for $i = 1, 2, \dots$ set $a_i = \sum_{j=1}^{\infty} \varphi_{ij}^2$. The sequence $\{a_i\}_{i=1}^{\infty}$ is absolutely summable, so applying Lemma 3.2 we obtain sequences $\{b_i\}_{i=1}^{\infty}$ and $\{c_i\}_{i=1}^{\infty}$ for which $a_i = b_i c_i$, $i = 1, 2, \dots$, $\lim_{i \rightarrow \infty} b_i = 0$, and $\sum_{i=1}^{\infty} |c_i| = \sum_{i=1}^{\infty} c_i < \infty$. Define $\Phi^1 \in \mathcal{L}(Y)$ by $\Phi^1 y = \sum_{i=1}^{\infty} \sqrt{b_i} \langle y, y_i \rangle_Y y_i$, $y \in Y$, and $\Phi^2 \in \mathcal{L}(X, Y)$ by $\Phi^2 x = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\varphi_{ij}}{\sqrt{b_i}} \langle x, x_j \rangle_X y_i$, $x \in X$. Then $\Phi^1 \Phi^2 = \Phi$, and, since $\lim_{i \rightarrow \infty} \sqrt{b_i} = 0$ and $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{\varphi_{ij}}{\sqrt{b_i}} \right)^2 = \sum_{i=1}^{\infty} \frac{1}{b_i} \sum_{j=1}^{\infty} \varphi_{ij}^2 = \sum_{i=1}^{\infty} \frac{a_i}{b_i} = \sum_{i=1}^{\infty} c_i < \infty$, it follows that Φ^1 is compact and $\Phi^2 \in HS(X, Y)$.

Lemma 3.4

- (a) For every $\Phi \in \mathcal{H}$, $\lim_{n \rightarrow \infty} |P_n \Phi P_n - \Phi|_{\mathcal{H}} = 0$.
- (b) For every $\Phi \in \mathcal{V}$, $\lim_{n \rightarrow \infty} \|P_n \Phi P_n - \Phi\|_{\mathcal{V}} = 0$.

Proof (a). For $\Phi \in \mathcal{H}$ we have

$$\begin{aligned} |P_n \Phi P_n - \Phi|_{\mathcal{H}} &\leq |P_n \Phi P_n - P_n \Phi|_{\mathcal{H}} + |P_n \Phi - \Phi|_{\mathcal{H}} \\ &\leq |\Phi P_n - \Phi|_{\mathcal{H}} + |P_n \Phi - \Phi|_{\mathcal{H}} \\ &= |(\Phi P_n)^* - \Phi^*|_{\mathcal{H}} + |P_n \Phi - \Phi|_{\mathcal{H}} \\ &= |P_n \Phi^* - \Phi^*|_{\mathcal{H}} + |P_n \Phi - \Phi|_{\mathcal{H}} \end{aligned}$$

where in the above estimate we have used the fact that $P_n \in \mathcal{L}(H)$ with $|P_n| \leq 1$ and that $|P_n \Psi|_{\mathcal{H}} \leq |P_n| |\Psi|_{\mathcal{H}} \leq |\Psi|_{\mathcal{H}}$ for every $\Psi \in \mathcal{H}$. If we apply Lemma 3.3 with $X = Y = H$ to $\Phi, \Phi^* \in \mathcal{H} = HS(H, H)$, then we obtain $\Phi = \Phi^1 \Phi^2$, $\Phi^* = (\Phi^*)^1 (\Phi^*)^2$ with $\Phi^1, (\Phi^*)^1 \in \mathcal{L}(H)$ compact and $(\Phi^*)^2 \in HS(H, H)$. It follows that

$$|P_n \Phi^* - \Phi^*|_{\mathcal{H}} = |(P_n - I)(\Phi^*)^1 (\Phi^*)^2|_{\mathcal{H}} \leq |(P_n - I)(\Phi^*)^1| |(\Phi^*)^2|_{\mathcal{H}},$$

and

$$|P_n \Phi - \Phi|_{\mathcal{H}} \leq |(P_n - I)\Phi^1| |\Phi^2|_{\mathcal{H}},$$

which together with assumption (3.1) and the fact that Φ^1 and $(\Phi^*)^1$ are compact yield the desired result.

(b). For $\Phi \in \mathcal{V} = HS(V^*, H) \cap HS(H, V)$ we have

$$\begin{aligned} |P_n \Phi P_n - \Phi|_{HS(V^*, H)} &\leq |P_n \Phi P_n - P_n \Phi|_{HS(V^*, H)} + |P_n \Phi - \Phi|_{HS(V^*, H)} \\ &\leq |P_n| |\Phi P_n - \Phi|_{HS(V^*, H)} + |P_n \Phi - \Phi|_{HS(V^*, H)} \\ &\leq |\Phi P_n - \Phi|_{HS(V^*, H)} + |P_n \Phi - \Phi|_{HS(V^*, H)}. \end{aligned}$$

Now $\Phi \in HS(V^*, H)$ implies that $\Phi^* \in HS(H, V^*)$ and that $(\Phi P_n)^* = P_n^* \Phi^* \in HS(H, V^*)$ where, recalling Lemma 3.1, $P_n^* = \gamma P_n \gamma^{-1}$ is the adjoint of the operator P_n considered as an element of $\mathcal{L}(V^*)$. It follows that

$$(3.4) \quad |P_n \Phi P_n - \Phi|_{HS(V^*, H)} \leq |P_n^* \Phi^* - \Phi^*|_{HS(H, V^*)} + |P_n \Phi - \Phi|_{HS(V^*, H)}$$

Lemma 3.3 with $X = H$ and $Y = V^*$ together with Lemma 3.1 imply that the first term on the right hand side of the estimate (3.4) tends to zero as $n \rightarrow \infty$. Similarly, Lemma 3.3 with $X = V^*$ and $Y = H$ together with assumption (3.1) imply that the second term tends to zero as $n \rightarrow \infty$ as well. A similar argument to the one given above can be used to show that $\lim_{n \rightarrow \infty} |P_n \Phi P_n - \Phi|_{HS(H, V)} = 0$ and the lemma is proved.

The primary result of this paper is given in the following theorem.

Theorem 3.1 Let $\Pi \in \mathcal{V}$ be the unique solution to the equation (2.5) (equivalently (2.3)) and for each $n = 1, 2, \dots$ let $\Pi_n \in \mathcal{H}_n$ be the unique solution to the approximating operator equation (3.3). Then $\lim_{n \rightarrow \infty} \|\Pi_n - \Pi\|_{\mathcal{V}} = 0$.

Proof From the triangle inequality we obtain

$$\|\Pi_n - \Pi\|_{\mathcal{V}} \leq \|\Pi_n - P_n \Pi P_n\|_{\mathcal{V}} + \|P_n \Pi P_n - \Pi\|_{\mathcal{V}}.$$

An application of Lemma 3.4(b) yields that the second term on the right and side of the above estimate tends to zero as $n \rightarrow \infty$. As for the first term, we recall (2.4) and consider the estimate

$$\begin{aligned} \omega \|\Pi_n - P_n \Pi P_n\|_{\mathcal{V}}^2 &\leq [\mathcal{A}\{\Pi_n - P_n \Pi P_n\}, \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &= [\mathcal{A}\Pi_n + \mathcal{B}_n(\Pi_n) - \mathcal{A}\Pi - \mathcal{B}(\Pi), \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad + [\mathcal{A}\Pi - \mathcal{A}P_n \Pi P_n, \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad + [\mathcal{B}(\Pi) - \mathcal{B}(P_n \Pi P_n), \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad + [\mathcal{B}(P_n \Pi P_n) - \mathcal{B}_n(\Pi_n), \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &= [\Theta_n - \Theta, \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad + [\mathcal{A}\{\Pi - P_n \Pi P_n\}, \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad + [\mathcal{B}(\Pi) - \mathcal{B}(P_n \Pi P_n), \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad - [P_n \mathcal{B}(\Pi_n) P_n - \mathcal{B}(P_n \Pi P_n), \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &= [P_n \Theta P_n - \Theta, \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad + [\mathcal{A}\{\Pi - P_n \Pi P_n\}, \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad + [\mathcal{B}(\Pi) - \mathcal{B}(P_n \Pi P_n), \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad - [\mathcal{B}(\Pi_n) - \mathcal{B}(P_n \Pi P_n), \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\leq [P_n \Theta P_n - \Theta, \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \\ &\quad + [\mathcal{A}\{\Pi - P_n \Pi P_n\}, \Pi_n - P_n \Pi P_n]_{\mathcal{H}} \end{aligned}$$

$$+[\mathcal{B}(\Pi) - \mathcal{B}(P_n \Pi P_n), \Pi_n - P_n \Pi P_n]_{\mathcal{H}}$$

where in the final estimate above we have applied assumption (2.1). Continuing we find that

$$\begin{aligned} \omega \|\Pi_n - P_n \Pi P_n\|_{\mathcal{V}}^2 &\leq \|P_n \Theta P_n - \Theta\|_{\mathcal{V}^*} \|\Pi_n - P_n \Pi P_n\|_{\mathcal{V}} \\ &\quad + |\mathcal{A}|_{\mathcal{L}(\mathcal{V}, \mathcal{V}^*)} \|\Pi - P_n \Pi P_n\|_{\mathcal{V}} \|\Pi_n - P_n \Pi P_n\|_{\mathcal{V}} \\ &\quad + \|\mathcal{B}(\Pi) - \mathcal{B}(P_n \Pi P_n)\|_{\mathcal{V}^*} \|\Pi_n - P_n \Pi P_n\|_{\mathcal{V}}, \end{aligned}$$

or, recalling the continuous embedding of H into V^* , that

$$\begin{aligned} \|\Pi_n - P_n \Pi P_n\|_{\mathcal{V}} &\leq K \|P_n \Theta P_n - \Theta\|_{\mathcal{H}} + |\mathcal{A}|_{\mathcal{L}(\mathcal{V}, \mathcal{V}^*)} \|\Pi - P_n \Pi P_n\|_{\mathcal{V}} \\ &\quad + K \|\mathcal{B}(\Pi) - \mathcal{B}(P_n \Pi P_n)\|_{\mathcal{H}}. \end{aligned}$$

Lemma 3.4 together with the continuity assumption on \mathcal{B} imply

$$\lim_{n \rightarrow \infty} \|\Pi_n - P_n \Pi P_n\|_{\mathcal{V}} = 0$$

and the theorem is established.

4. AN EXAMPLE

In order to illustrate the application of our theory we consider an operator algebraic Riccati equation arising in the design of an optimal feedback control law for a one dimensional heat/diffusion equation. Let $H, U = L_2(0, 1)$, both endowed with the usual inner product, $(\varphi, \psi) = \int_0^1 \varphi(\eta) \psi(\eta) d\eta$, and consider the linear-quadratic optimal control problem of finding $\bar{u} \in L_2(0, \infty; U)$ which minimizes the quadratic performance index

$$J(u) = \int_0^\infty (Qx(t, \cdot), x(t, \cdot)) + r(u(t, \cdot), u(t, \cdot)) dt$$

subject to the linear dynamical system

$$(4.1) \quad \frac{\partial x}{\partial t}(t, \eta) - \frac{\partial}{\partial \eta} a(\eta) \frac{\partial x}{\partial \eta}(t, \eta) = bu(t, \eta), t > 0, 0 < \eta < 1,$$

$$(4.2) \quad x(t, 0) = 0, x(t, 1) = 0, t > 0$$

$$(4.3) \quad x(0, \eta) = x^0(\eta), 0 < \eta < 1,$$

where Q is a self-adjoint, nonnegative, and Hilbert-Schmidt operator from $L_2(0, 1)$ into $L_2(0, 1)$, $a \in L_\infty(0, 1)$, $a(\eta) \geq \alpha > 0$, a.e. $\eta \in (0, 1)$, $b \in \mathbb{R}$, and $x^0 \in L_2(0, 1)$. If we define $\mathcal{C} = \{\Phi \in HS(L_2(0, 1), L_2(0, 1)), \Phi = \Phi^*, \Phi \geq 0\}$, then $Q \in \mathcal{C}$ and $(Q\varphi)(\eta) = \int_0^1 q(\eta, \zeta) \varphi(\zeta) d\zeta$ with $q \in L_2((0, 1) \times (0, 1))$, $q(\eta, \zeta) = q(\zeta, \eta)$, $q(\eta, \zeta) \geq 0$, a.e. $(\eta, \zeta) \in (0, 1) \times (0, 1)$.

Define $V = H_0^1(0, 1)$ endowed with the standard inner product, $\langle \varphi, \psi \rangle = \int_0^1 D\varphi(\eta) D\psi(\eta) d\eta$ and corresponding norm, $\|\cdot\|$. It follows that V is densely, continuously, and compactly embedded in H , that $V^* = H^{-1}(0, 1)$, and that H is

densely and continuously embedded in V^* . Define the operator $A \in \mathcal{L}(V, V^*)$ by $(A\varphi, \psi) = (aD\varphi, D\psi)$, for $\varphi, \psi \in \mathcal{V}$. It follows that $(A\varphi, \varphi) \geq \alpha\|\varphi\|^2$, $\varphi \in V$, and that the restriction \tilde{A} of the operator $-A$ to the set $Dom(\tilde{A}) = \{\varphi \in V : A\varphi \in H\}$ ($= H^2(0, 1) \cap H_0^1(0, 1)$ when a is sufficiently smooth) is densely defined in H , negative, self-adjoint, and it is the infinitesimal generator of a uniformly exponentially stable analytic semigroup, $\{T(t) : t \geq 0\}$ of bounded, self-adjoint linear operators on H . The solution to the initial-boundary value problem (4.1)-(4.3) is given by

$$x(t) = T(t)x^0 + \int_0^t T(t-s)bu(s)ds, \quad t > 0$$

where for each $t > 0$ $x(t) = x(t, \cdot) \in H = L_2(0, 1)$ and $u(t) = u(t, \cdot) \in U = L_2(0, 1)$ for almost every $t > 0$.

The solution to the optimal control problem (see [5]) is given in closed-loop linear state feedback form by

$$\bar{u}(t) = -(b/r)\Pi x(t), \quad a.e. \ t > 0$$

where Π is the unique nonnegative self-adjoint solution to the algebraic Riccati equation

$$(4.4) \quad A^*\Pi + \Pi A + (b^2/r)\Pi^2 = Q.$$

It is clear that the existence-uniqueness and approximation theories presented above apply with $\Theta = Q \in \mathcal{C}$ and $\mathcal{B}(\Phi) = (b^2/r)\Phi^2$ for $\Phi \in \mathcal{C}$. It follows that there exists a unique solution $\Pi \in HS(L_2(0, 1), L_2(0, 1))$ to the nonlinear operator equation (4.4) with $\Pi = \Pi^*$, $\Pi \geq 0$, and $\Pi \in HS(L_2(0, 1), H_0^1(0, 1)) \cap HS(H^{-1}(0, 1), L_2(0, 1))$. Recalling that $HS(L_2(0, 1), L_2(0, 1))$ is isometrically isomorphic to $L_2((0, 1) \times (0, 1))$, $\Pi \in \mathcal{C}$ implies that there exists $\pi \in L_2((0, 1) \times (0, 1))$ with $\pi(\eta, \zeta) = \pi(\zeta, \eta)$ and $\pi(\eta, \zeta) \geq 0$ for almost every $(\eta, \zeta) \in (0, 1) \times (0, 1)$ for which

$$\bar{u}(t, \eta) = -(b/r) \int_0^1 \pi(\eta, \zeta)x(t, \zeta)d\zeta,$$

for almost every $\eta \in (0, 1)$ and $t > 0$.

We introduce linear spline based approximation. For each $n=2,3,\dots$ let $H_n = span \{\varphi_n^j\}_{j=1}^{n-1}$ where for $j=1,2,\dots,n-1$, φ_n^j denotes the j th standard linear spline function defined on the interval $[0,1]$ with respect to the uniform mesh $\{0, 1/n, 2/n, \dots, 1\}$. More precisely,

$$\varphi_n^j(\eta) = \begin{cases} 0 & 0 \leq \eta \leq j^{-1}/n \\ n\eta - j + 1 & j^{-1} \leq \eta \leq j/n \\ j + 1 - n\eta & j/n \leq \eta \leq j + 1/n \\ 0 & j + 1/n \leq \eta \leq 1, \end{cases}$$

$j=1,2,\dots,n-1$. Let $P_n : H \rightarrow H_n$ denote the orthogonal projection of H onto H_n with respect to the usual inner product on $H = L_2(0,1)$ and define Galerkin approximations $A_n \in \mathcal{L}(H_n)$ to A in the usual way. That is let $A_n \varphi_n = \psi_n$ where for $\varphi_n \in H_n$, ψ_n is the unique element in H_n which satisfies $(A\varphi_n, \chi_n) = (\psi_n, \chi_n)$, $\chi_n \in H_n$. Set $Q_n = P_n Q \in \mathcal{L}(H_n)$.

Using well known approximation properties of linear interpolatory splines (see [7]) it is not difficult to argue that $\lim_{n \rightarrow \infty} \|P_n \varphi - \varphi\| = 0$, $\varphi \in H_0^1(0,1)$ and consequently that assumption (3.1) is satisfied. It follows therefore, from the theory presented in section 3 above, that there exists a unique nonnegative self-adjoint operator $\Pi_n \in \mathcal{L}(H_n)$ satisfying the algebraic Riccati equation in H_n given by

$$(4.5) \quad A_n^* \Pi_n + \Pi_n A + (b^2/r) \Pi_n^2 = Q_n.$$

Moreover, we have the Hilbert-Schmidt norm convergence of $\Pi_n P_n$ to Π as $n \rightarrow \infty$. That is

$$(4.6) \quad \lim_{n \rightarrow \infty} |\Pi_n P_n - \Pi|_{HS(H,H)} = 0.$$

We in fact also obtain that $\lim_{n \rightarrow \infty} |\Pi_n P_n - \Pi|_{HS(H,V)} = 0$ and that $\lim_{n \rightarrow \infty} |\Pi_n P_n - \Pi|_{HS(V^*,H)} = 0$.

From a computational point of view, since the basis elements φ_n^j are not mutually orthonormal, simply replacing the operators in the finite dimensional algebraic Riccati equation (4.5) with their corresponding matrix representations will not lead to the usual symmetric matrix Riccati equation for which a variety of computational solution techniques exist. Toward this end, for a linear operator L_n with domain and/or range in H_n , we denote its matrix representation with respect to the basis $\{\varphi_n^j\}_{j=1}^{n-1}$ for H_n by L_N . Define $\phi_n : [0,1] \rightarrow \mathbb{R}^{n-1}$ by $\phi_n(\eta) = (\varphi_n^1(\eta), \varphi_n^2(\eta), \dots, \varphi_n^{n-1}(\eta))^T$ and set $M_N = (\phi_n, \phi_n^T) = \int_0^1 \phi_n(\eta) \phi_n^T(\eta) d\eta$. It then follows that $A_N = M_N^{-1}(aD\phi_n, D\phi_n^T)$ with $A_N^* = M_N^{-1}A_N^T M_N$, and $Q_N = M_N^{-1}(Q\phi_n, \phi_n^T)$. If we set $\hat{Q}_N = M_N Q_N$ and $\hat{\Pi}_N = M_N \Pi_N$, then $\hat{\Pi}_N$ is the unique nonnegative self-adjoint solution to the $(n-1) \times (n-1)$ matrix algebraic Riccati equation given by

$$(4.7) \quad A_N^T \hat{\Pi}_N + \hat{\Pi}_N A_N + (b^2/r) \hat{\Pi}_N M_N^{-1} \hat{\Pi}_N = \hat{Q}_N.$$

The approximating optimal control laws take the form

$$\bar{u}_n(t, \eta) = -(b/r) \int_0^1 \pi_n(\eta, \zeta) x(t, \zeta) d\zeta$$

for almost every $\eta \in (0, 1)$ and $t > 0$ where

$$\pi_n(\eta, \zeta) = \phi_n(\eta)^T M_N^{-1} \hat{\Pi}_N M_N^{-1} \phi_n(\zeta),$$

$(\eta, \zeta) \in [0, 1] \times [0, 1]$. It follows from (4.6) that $\lim_{n \rightarrow \infty} \pi_n = \pi$ in $L_2((0, 1) \times (0, 1))$ - that is that

$$(4.8) \quad \lim_{n \rightarrow \infty} \int_0^1 \int_0^1 |\pi_n(\eta, \zeta) - \pi(\eta, \zeta)|^2 d\zeta d\eta.$$

To illustrate we take $a(\eta) = a > 0$, a constant, and let $Q \in HS(L_2(0, 1), L_2(0, 1))$ be the finite rank modal projection operator given by

$$Q\varphi = \sum_{k=1}^{\nu} q_k(\varphi, \psi_k) \psi_k, \quad \varphi \in L_2(0, 1)$$

where $\nu < \infty$, $\psi_k(\eta) = \sqrt{2} \sin k\pi\eta$, $\eta \in [0, 1]$, $k = 1, 2, \dots, \nu$, and $q_k \geq 0$, $k = 1, 2, \dots, \nu$. A somewhat tedious, but rather straight forward computation yields

$$(\hat{Q}_N)_{ij} = \sum_{k=1}^{\nu} 2q_k \delta_{ki}^n \delta_{kj}^n, \quad i, j = 1, 2, \dots, n-1,$$

where

$$\delta_{k\ell}^n = \frac{-n}{(k\pi)^2} \left\{ \sin k \frac{\pi(\ell+1)}{n} - 2 \sin k \frac{\pi\ell}{n} + \sin k \frac{\pi(\ell-1)}{n} \right\},$$

$k = 1, 2, \dots, \nu$, $\ell = 1, 2, \dots, n-1$. Setting $a = .25$, $b = 1.0$, $r = .01$, $\nu = 3$, and $q_1 = q_2 = q_3 = 1.0$, and using Schur-vector decomposition of the associated Hamiltonian matrix (see [4]) to solve the matrix Riccati equation (4.7) for various values of n we obtained the kernels, π_n , plotted in the figures below. That the convergence given in (4.8) above is achieved is immediately clear.

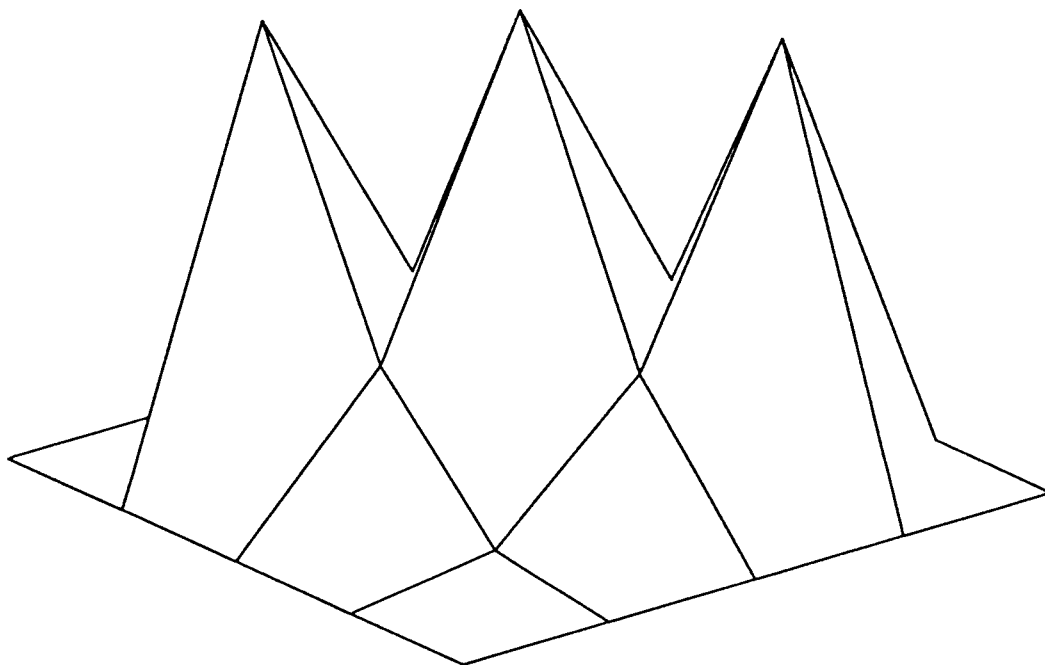


Figure 4.1a: $\pi_4(\eta, \zeta)$, $(\eta, \zeta) \in [0, 1] \times [0, 1]$.

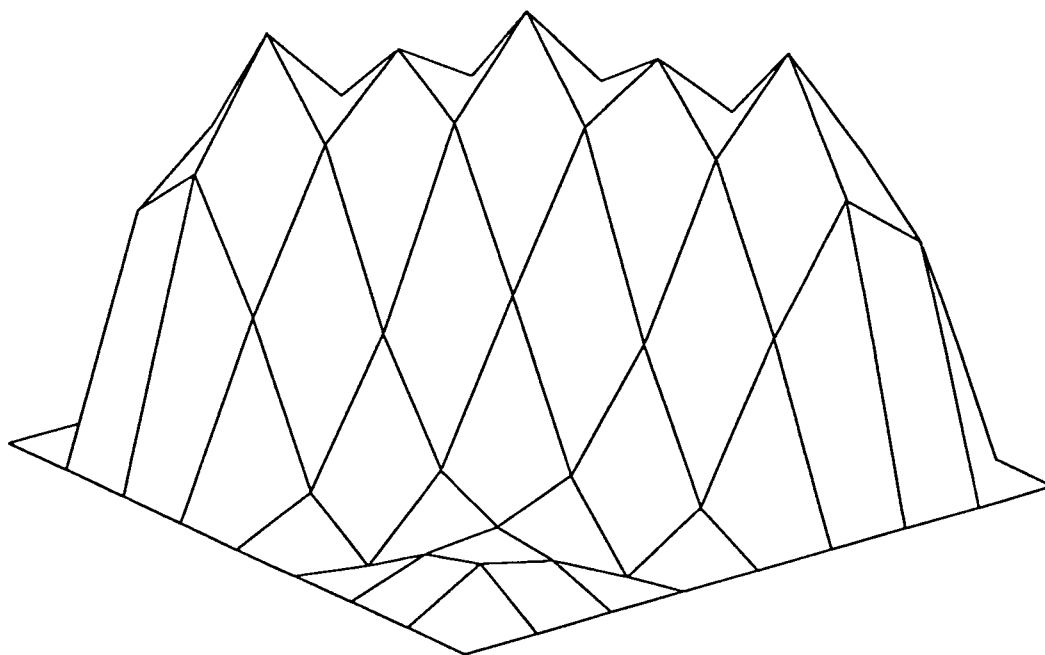


Figure 4.1b: $\pi_8(\eta, \zeta)$, $(\eta, \zeta) \in [0, 1] \times [0, 1]$.

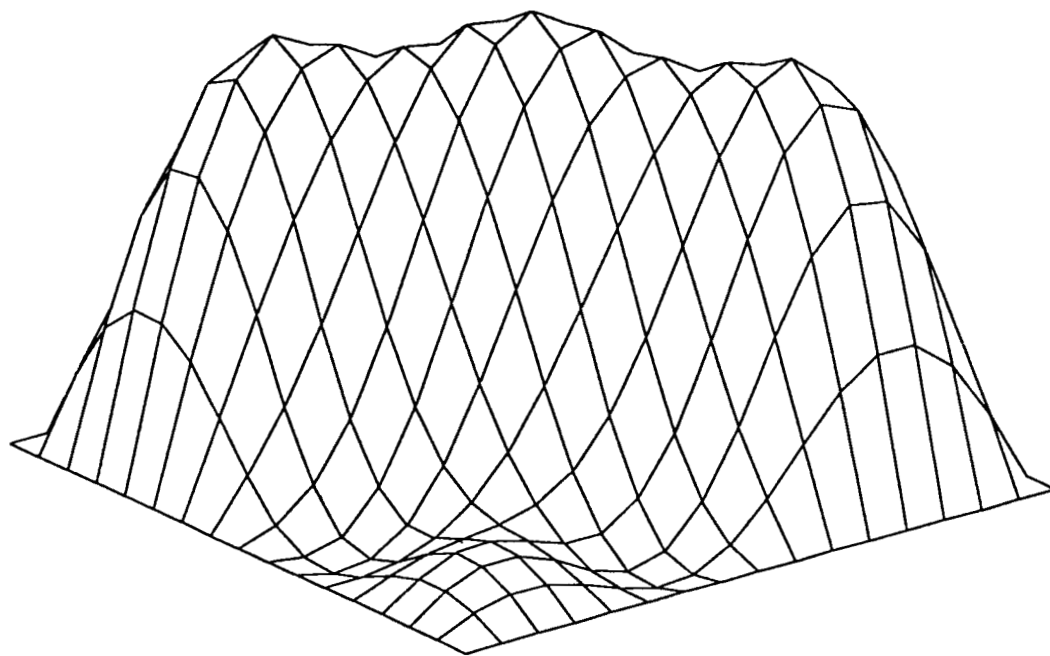


Figure 4.1c: $\pi_{16}(\eta, \zeta)$, $(\eta, \zeta) \in [0, 1] \times [0, 1]$.

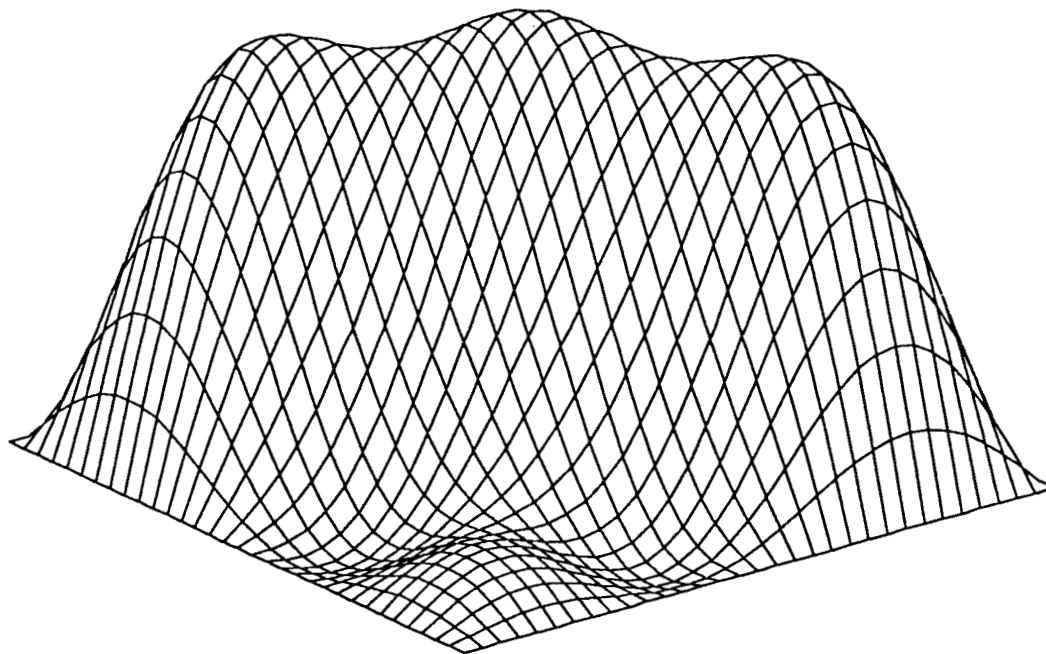


Figure 4.1d: $\pi_{32}(\eta, \zeta)$, $(\eta, \zeta) \in [0, 1] \times [0, 1]$.

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